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On strong and total Lagrange duality for convex optimization problems

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Abstract

We give some necessary and sufficient conditions which completely characterize the strong and total Lagrange duality, respectively, for convex optimization problems in separated locally convex spaces. We also prove similar statements for the problems obtained by perturbing the objective functions of the primal problems by arbitrary linear functionals. In the particular case when we deal with convex optimization problems having infinitely many convex inequalities as constraints the conditions we work with turn into the so-called Farkas–Minkowski and locally Farkas–Minkowski conditions for systems of convex inequalities, recently used in the literature. Moreover, we show that our new results extend some existing ones in the literature.

Keywords: Conjugate functions; Lagrange dual problem; Basic constraint qualification; (Locally) Farkas–Minkowski condition; Stable strong duality

1. Introduction

Consider a convex optimization problem

$$\inf_{\substack{x\in U,\\ g(x)\in -C}} f(x), \tag{P}$$

where X and Y are separated locally convex vector spaces, U is a non-empty closed convex subset of X, C is a non-empty closed convex cone in Y, $f : X \to \overline{\mathbb{R}}$ is a proper convex lower semicontinuous function and $g : X \to Y^\bullet$ is a proper C-convex function. Moreover, take g to be C-epi-closed, i.e. its C-epigraph $\operatorname{epi}_C(g) = \{(x, y) \in X \times Y\colon\ y \in g(x) + C\}$ is a closed set. C-epi-closedness is an extension of the lower semicontinuity for vector functions which is more general than the ones usually used in optimization. The Lagrange dual problem to (P) is

$$\sup_{\lambda\in C^*} \inf_{x\in U} \big[f(x) + (\lambda g)(x)\big]. \tag{D}$$

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In order to have *strong duality* for (P) and (D), i.e. the situation when their optimal objective values coincide and the dual has an optimal solution, we gave in [3] (see also [2]) a constraint qualification of closedness-type which is weaker than the classical interior point conditions considered so far in the literature. Moreover, we have completely characterized through equivalent conditions the strong duality for the problems obtained by perturbing the objective function of (P) by arbitrary linear functionals and their Lagrange duals. The situation when strong duality holds for all the problems obtained by linearly perturbing the objective function of (P) and their corresponding duals is called *stable strong duality*.

Motivated by [11], we give in this paper a condition which completely characterizes the strong Lagrange duality for all the optimization problems (P) whose objective functions satisfy a certain regularity condition (cf. [9,11]). Similar characterizations are given also for all the optimization problems (P) for which the existence of an optimal solution is assumed and whose objective functions satisfy a weak condition (see [4,5]). In this case the strong duality will be named *total duality*. This notion has been called so in order to underline the fact that both the primal and the dual problems have optimal solutions and their optimal objective values coincide.

We also introduce a new condition that completely characterizes the *stable total Lagrange duality*, namely the situation when there is strong duality for all the problems obtained by linearly perturbing the objective function of (P) for which an optimal solution exists and their Lagrange duals. Moreover, when the condition that completely characterizes the strong Lagrange duality for (P) is fulfilled, we give optimality conditions for (P) by using subdifferentials.

Particularizing the cone constraints in (P) by considering the order induced by the cone $\mathbb{R}^T_+$, where T is an arbitrary index set, we rediscover and sometimes improve some results given in recent works dealing with characterizations of systems of infinitely many convex inequalities (see [9–11]). The conditions we consider become then the *Farkas–Minkowski* and *locally Farkas–Minkowski* conditions used in the mentioned papers. In some situations the locally Farkas–Minkowski condition turns out to be equivalent to the celebrated *basic constraint qualification* (*BCQ*) introduced first in [13], treated also in [14,17,18,21].

The paper is organized as follows. Section 2 is dedicated to the necessary preliminaries in order to make the paper self-contained. In Section 3 we consider the condition that completely characterizes the strong Lagrange duality for all the optimization problems (P) whose objective functions satisfy a closedness condition, mentioning, where is the case, which results from the literature are rediscovered as special cases and improved. Section 4 is dedicated to similar characterizations for optimization problems (P) with the objective functions satisfying some weak conditions and for which the existence of optimal solutions is guaranteed. Optimality conditions for such problems are also given via subdifferentials. A short conclusive section closes the paper.

2. Preliminaries

Consider two separated locally convex vector spaces X and Y and their continuous dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$, respectively. Let the non-empty closed convex cone $C \subseteq Y$ and its dual cone $C^* = \{y^* \in Y^*\colon \langle y^*, y\rangle \geqslant 0\ \forall y \in Y\}$ be given, where we denote by $\langle y^*, y\rangle = y^*(y)$ the value at y of the continuous linear functional y^*. On Y we consider the partial order induced by C, "$\leqslant_C$," defined by $z \leqslant_C y \Leftrightarrow y - z \in C$, $z, y \in Y$. To Y we attach a greatest element with respect to "$\leqslant_C$" denoted by ∞_Y which does not belong to Y and let $Y^\bullet = Y \cup \{\infty_Y\}$. Then for any $y \in Y^\bullet$ one has $y \leqslant_C \infty_Y$ and we consider on $Y^\bullet$ the following operations: $y + \infty_Y = \infty_Y + y = \infty_Y$ and $t\infty_Y = \infty_Y$ for all $y \in Y$ and all $t \geqslant 0$. Denote also the set of non-negative real numbers by $\mathbb{R}_+ = [0, +\infty)$ and the cardinality of a set T by $\operatorname{card}(T)$.

Given a subset U of X, by $\operatorname{cl}(U)$ we denote its *closure* in the corresponding topology, by $\operatorname{bd}(U)$ its *boundary*, while its *indicator* function $\delta_U : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and, respectively, *support* function $\sigma_U : X^* \to \overline{\mathbb{R}}$ are defined as follows

$$\delta_U(x) = \begin{cases} 0, & \text{if } x \in U, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and} \quad \sigma_U(x^*) = \sup_{x \in U} \langle x^*, x\rangle.$$

Next we give some notions regarding functions.

For a function $f : X \to \overline{\mathbb{R}}$ we have

- the *domain*: $\operatorname{dom}(f) = \{x \in X\colon f(x) < +\infty\}$,
- the *epigraph*: $\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R}\colon f(x) \leqslant r\}$,

- the *conjugate regarding the set* $U \subseteq X$: $f_U^*: X^* \to \overline{\mathbb{R}}$ given by $f_U^*(x^*) = \sup\{\langle x^*, x\rangle - f(x): x \in U\}$,
- f is *proper*: $f(x) > -\infty$ $\forall x \in X$ and $\mathrm{dom}(f) \neq \emptyset$,
- the *subdifferential* of f at x, where $f(x) \in \mathbb{R}$: $\partial f(x) = \{x^* \in X^*: f(u) - f(x) \geqslant \langle x^*, u - x\rangle\ \forall u \in X\}$.

One can easily notice that $\delta_U^* = \sigma_U$. When $U = X$ the conjugate regarding the set U is the classical (*Fenchel–Moreau*) *conjugate* function of f denoted by f^*. Between a function and its conjugate regarding some set $U \subseteq X$ *Young–Fenchel's inequality* holds

$$f_U^*(x^*) + f(x) \geqslant \langle x^*, x\rangle \quad \forall x \in U\ \forall x^* \in X^*.$$

Given any proper function $f: X \to \overline{\mathbb{R}}$, for some $x \in \mathrm{dom}(f)$ and $x^* \in X^*$ one has

$$x^* \in \partial f(x) \quad \Leftrightarrow \quad f^*(x^*) + f(x) = \langle x^*, x\rangle.$$

Given two proper functions $f, g: X \to \overline{\mathbb{R}}$, we have the *infimal convolution* of f and g defined by

$$f \,\square\, g: X \to \overline{\mathbb{R}}, \quad (f \,\square\, g)(a) = \inf\bigl\{f(x) + g(a - x): x \in X\bigr\},$$

which is called *exact* at some $a \in X$ when there is an $x \in X$ such that $(f \,\square\, g)(a) = f(x) + g(a - x)$.

There are notions given for functions with extended real values that can be formulated also for functions having their ranges in infinite dimensional spaces.

For a function $g: X \to Y^\bullet$ one has

- the *domain*: $\mathrm{dom}(g) = \{x \in X: g(x) \in Y\}$,
- g is *proper*: $\mathrm{dom}(g) \neq \emptyset$,
- g is C-*convex*: $g(tx + (1-t)y) \leqslant_C tg(x) + (1-t)g(y)$ $\forall x, y \in X$ $\forall t \in [0, 1]$,
- for $\lambda \in C^*$, $(\lambda g): X \to \overline{\mathbb{R}}$, $(\lambda g)(x) = \langle \lambda, g(x)\rangle$ for $x \in \mathrm{dom}(g)$ and $(\lambda g)(x) = +\infty$ otherwise,
- the C-*epigraph*: $\mathrm{epi}_C(g) = \{(x, y) \in X \times Y: y \in g(x) + C\}$,
- g is C-epi-*closed*: $\mathrm{epi}_C(g)$ is closed,
- g is *star* C-*lower-semicontinuous* at $x \in X$: (λg) is lower-semicontinuous at x $\forall \lambda \in C^*$,
- for a subset $W \subseteq Y$: $g^{-1}(W) = \{x \in X: \exists z \in W \text{ s.t. } g(x) = z\}$.

Remark 1. Besides the two generalizations of lower semicontinuity (see [1]) defined above for functions taking values in infinite dimensional spaces in convex optimization there is widely used in the literature also the C-lower semicontinuity, introduced in [20] and refined in [7]. It was shown (see [19], for instance) that C-lower semicontinuity implies the star C-lower semicontinuity, which yields C-epi-closedness, while the opposite assertions are valid only under additional hypotheses. There are functions which have one of these properties, but not the stronger ones, see for instance the example in [20], where a C-epi-closed function which is not C-lower semicontinuous is given. Unfortunately that function is not C-convex. We give below a C-convex function which is C-epi-closed, but not star C-lower semicontinuous. Although most of the research related to what we present in this paper is performed by considering the stronger types of generalized lower semicontinuous vector functions, we work here in the most general framework.

Example 1. Consider the function

$$g: \mathbb{R} \to \bigl(\mathbb{R}^2\bigr)^\bullet = \mathbb{R}^2 \cup \{\infty\}, \quad g(x) = \begin{cases} (\frac{1}{x}, x), & \text{if } x > 0,\\ \infty, & \text{otherwise.}\end{cases}$$

One can show that g is $\mathbb{R}^2_+$-convex and $\mathbb{R}^2_+$-epi-closed, but not star $\mathbb{R}^2_+$-lower semicontinuous. For instance, for $\lambda = (0, 1)^T \in (\mathbb{R}^2_+)^* = \mathbb{R}^2_+$ one has

$$\bigl((0, 1)^T g\bigr)(x) = \begin{cases} x, & \text{if } x > 0,\\ +\infty, & \text{otherwise,}\end{cases}$$

which is not lower semicontinuous.

The following statement was proven in [15] and then in [16] under the assumption of continuity, respectively star C-lower semicontinuity for the function involved. We extend it by considering the function g C-epi-closed.

Lemma 1. *Let $U \subseteq X$ a non-empty closed convex set and a proper, C-convex and C-*epi-*closed function $g : X \to Y^\bullet$ such that $U \cap g^{-1}(-C) \neq \emptyset$. Then*

$$\operatorname{epi}(\sigma_{U\cap g^{-1}(-C)}) = \operatorname{cl}\biggl(\operatorname{epi}(\sigma_U) + \bigcup_{\lambda\in C^*} \operatorname{epi}\bigl((\lambda g)^*\bigr)\biggr).$$

Proof. Consider the functions $F, G : Y \times X \to \overline{\mathbb{R}}$, defined by $F(y,x) = \delta_{\{0\}\times U}(y,x)$ and, respectively, $G(y,x) = \delta_{\{(y,x)\in Y\times X:\ g(x)-y\in -C\}}(y,x)$. Both these functions are proper, convex and lower semicontinuous, thus applying Theorem 2.1 in [4] we get $\operatorname{epi}((F+G)^*) = \operatorname{cl}(\operatorname{epi}(F^*) + \operatorname{epi}(G^*))$.

Simple calculations show that $\operatorname{epi}(F^*) = Y^* \times \operatorname{epi}(\sigma_U)$ and $\operatorname{epi}(G^*) = \bigcup_{\lambda\in C^*}\{(-\lambda, p, r):\ (p,r) \in \operatorname{epi}((\lambda g)^*)\}$, thus

$$\begin{aligned}\operatorname{epi}\bigl((F+G)^*\bigr) &= \operatorname{cl}\biggl(Y^* \times \biggl(\operatorname{epi}(\sigma_U) + \bigcup_{\lambda\in C^*}\bigl\{(p,r):\ (p,r)\in \operatorname{epi}\bigl((\lambda g)^*\bigr)\bigr\}\biggr)\biggr)\\ &= Y^* \times \operatorname{cl}\biggl(\operatorname{epi}(\sigma_U) + \bigcup_{\lambda\in C^*}\operatorname{epi}\bigl((\lambda g)^*\bigr)\biggr).\end{aligned}$$

On the other hand, it is not difficult to notice that for all $(y,x) \in Y \times X$ there is

$$F(y,x) + G(y,x) = \delta_{\{0\}\times(U\cap g^{-1}(-C))}(y,x).$$

Then the epigraph of $(F+G)^*$ coincides with $Y^* \times \operatorname{epi}(\sigma_{U\cap g^{-1}(-C)})$. Hence, we get

$$Y^* \times \operatorname{epi}(\sigma_{U\cap g^{-1}(-C)}) = Y^* \times \operatorname{cl}\biggl(\operatorname{epi}(\sigma_U) + \bigcup_{\lambda\in C^*}\operatorname{epi}\bigl((\lambda g)^*\bigr)\biggr),$$

which yields $\operatorname{epi}(\sigma_{U\cap g^{-1}(-C)}) = \operatorname{cl}(\operatorname{epi}(\sigma_U) + \bigcup_{\lambda\in C^*}\operatorname{epi}((\lambda g)^*))$. $\square$

From the general case we get as special cases some results previously given for semi-infinite systems of convex inequalities. This is the reason why we recall some notations used in the literature on semi-infinite programming. Let T be a possibly infinite index set and denote by $\mathbb{R}^T$ the space of all functions $x : T \to \mathbb{R}$, endowed with the product topology and with the operations being the usual pointwise ones. For simplicity, denote $x_t = x(t)\ \forall x \in \mathbb{R}^T\ \forall t \in T$. The dual space of $\mathbb{R}^T$ is $(\mathbb{R}^T)^*$, the *space of generalized finite sequences* $\lambda = (\lambda_t)_{t\in T}$ such that $\lambda_t \in \mathbb{R}\ \forall t \in T$, and with finitely many λ_t different from zero. The positive cone in $\mathbb{R}^T$ is $\mathbb{R}^T_+ = \{x \in \mathbb{R}^T:\ x_t = x(t) \geqslant 0\ \forall t \in T\}$, and its dual is the positive cone in $(\mathbb{R}^T)^*$, namely $(\mathbb{R}^T_+)^* = \{\lambda = (\lambda_t)_{t\in T} \in (\mathbb{R}^T)^*:\ \lambda_t \geqslant 0\ \forall t \in T\}$.

For a convex optimization problem (P) we denote by $v(P)$ its optimal objective value. Let us recall that by *strong duality* we understand the situation when the optimal objective values of the primal and dual problem coincide and the dual problem has an optimal solution. In the following we will write min (max) instead of inf (sup) when the infimum (supremum) is attained.

3. New characterizations for strong Lagrange duality

Consider the separated locally convex vector spaces X and Y. Let U be a non-empty closed convex subset of X, C a non-empty closed convex cone in Y and $g : X \to Y^\bullet$ a proper C-convex C-epi-closed function. Denote $\mathcal{A} = \{x \in U:\ g(x) \in -C\}$ and assume this set non-empty. By the assumptions we made it is clear that $\mathcal{A}$ is a convex and closed set. For a proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ fulfilling $\mathcal{A} \cap \operatorname{dom}(f) \neq \emptyset$ consider the optimization problem

$$\inf_{x\in\mathcal{A}} f(x). \tag{P}$$

The stable strong duality for this problem and its Lagrange dual is completely characterized through the following condition

$$\bigcup_{\lambda\in C^*}\operatorname{epi}\bigl(\bigl(f + (\lambda g) + \delta_U\bigr)^*\bigr) \quad \text{is closed.} \tag{$C(f,\mathcal{A})$}$$

Theorem 1. *The set $\mathcal{A}$ and the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ satisfy condition $(C(f,\mathcal{A}))$ if and only if for any $p \in X^*$ one has*

$$\inf_{\substack{x\in U,\\ g(x)\in -C}} \big[f(x) + \langle p, x\rangle\big] = \max_{\lambda\in C^*} \inf_{x\in U}\big[f(x) + (\lambda g)(x) + \langle p, x\rangle\big].$$

This statement has been proven in [3] for g C-epi-closed, while in [16] it was given under the stronger assumption that g is star C-lower semicontinuous.

When taking the function f to be equal to 0 everywhere, the condition $(C(f,\mathcal{A}))$ becomes

$$\bigcup_{\lambda\in C^*} \operatorname{epi}\big(\big((\lambda g) + \delta_U\big)^*\big) \quad \text{is closed}, \tag{$C(0,\mathcal{A})$}$$

which was called *dual CQ* in [16]. In the cited paper this condition was introduced as a weak constraint qualification which guarantees strong duality for the convex optimization problem (P) and its Lagrange dual (D). In [3] we gave a weaker constraint qualification that ensured strong duality for this pair of problems.

Remark 1. When g is continuous at some point of $\mathcal{A}$, $(C(0,\mathcal{A}))$ means actually that $\operatorname{epi}(\sigma_U) + \bigcup_{\lambda\in C^*} \operatorname{epi}((\lambda g)^*)$ is closed, a condition known as $(CCCQ)$ (see [5,8,11,15]).

In the following statement we completely characterize via $(C(0,\mathcal{A}))$ the strong duality for the problem of minimizing a linear continuous functional over $\mathcal{A}$ and its Lagrange dual problem. It is a consequence of the previous theorem, when one takes $f(x) = 0$ $\forall x \in X$. In the special case $g : X \to Y$ C-convex and continuous we rediscover Theorem 3.2 in [5].

Corollary 1. *$\mathcal{A}$ fulfills the condition $(C(0,\mathcal{A}))$ if and only if for each $p \in X^*$ one has*

$$\inf_{x\in\mathcal{A}} \langle p, x\rangle = \max_{\lambda\in C^*} \inf_{x\in U}\big[\langle p, x\rangle + (\lambda g)(x)\big].$$

Next we give a statement where the strong duality for a convex optimization problem consisting in minimizing over the set $\mathcal{A}$ a proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ which satisfies the feasibility condition $\mathcal{A} \cap \operatorname{dom}(f) \neq \emptyset$ and fulfills the following condition (cf. [6,9,11])

$$\operatorname{epi}(f^*) + \operatorname{epi}(\sigma_{\mathcal{A}}) \quad \text{is closed in the product topology of } \big(X^*, w(X^*, X)\big) \times \mathbb{R}, \tag{CC}$$

and its Lagrange dual problem is completely characterized via $(C(0,\mathcal{A}))$.

Remark 2. (See [4].) If one removes the assumption of lower semicontinuity from f and takes it continuous at some point of $\mathcal{A}$, then condition (CC) is automatically satisfied.

Theorem 2. *$\mathcal{A}$ fulfills the condition $(C(0,\mathcal{A}))$ if and only if for each proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ which satisfies $\mathcal{A} \cap \operatorname{dom}(f) \neq \emptyset$ and (CC) one has*

$$\inf_{x\in\mathcal{A}} f(x) = \max_{\lambda\in C^*} \inf_{x\in U}\big[f(x) + (\lambda g)(x)\big].$$

Proof. The sufficiency follows from the previous corollary by taking f linear and continuous. To prove the necessity take first a function f which fulfills the hypotheses. Denote by (P) the optimization problem of minimizing f over $\mathcal{A}$ and by (D) its Lagrange dual problem.

If $v(P) = -\infty$ we are done, because of the weak duality for (P) and (D). Otherwise we have $v(P) \in \mathbb{R}$. Then it is obvious that $(f + \delta_{\mathcal{A}})^*(0) = -v(P)$. Further, we have $(0, -v(P)) \in \operatorname{epi}((f + \delta_{\mathcal{A}})^*)$. Because of Theorem 2.1 in [4], (CC) means actually $\operatorname{epi}((f + \delta_{\mathcal{A}})^*) = \operatorname{epi}(f^*) + \operatorname{epi}(\sigma_{\mathcal{A}})$. As Lemma 1 yields

$$\operatorname{epi}(\sigma_{\mathcal{A}}) = \operatorname{cl}\Big(\operatorname{epi}(\sigma_U) + \bigcup_{\lambda\in C^*} \operatorname{epi}\big((\lambda g)^*\big)\Big) = \operatorname{cl}\Big(\bigcup_{\lambda\in C^*} \big(\operatorname{epi}(\sigma_U) + \operatorname{epi}\big((\lambda g)^*\big)\big)\Big)$$

and standard calculations show that $\operatorname{epi}(\sigma_U)+\operatorname{epi}((\lambda g)^*)\subseteq \operatorname{epi}((\lambda g)^*_U)$ $\forall \lambda\in C^*$, we have

$$\operatorname{epi}(\sigma_{\mathcal{A}})\subseteq \operatorname{cl}\biggl(\bigcup_{\lambda\in C^*}\operatorname{epi}\bigl((\lambda g)^*_U\bigr)\biggr)=\bigcup_{\lambda\in C^*}\operatorname{epi}\bigl((\lambda g)^*_U\bigr),$$

the latter equality following from $(C(0,\mathcal{A}))$. Consequently,

$$\operatorname{epi}\bigl((f+\delta_{\mathcal{A}})^*\bigr)\subseteq \operatorname{epi}(f^*)+\bigcup_{\lambda\in C^*}\operatorname{epi}\bigl((\lambda g)^*_U\bigr)=\operatorname{epi}(f^*)+\bigcup_{\lambda\in C^*}\operatorname{epi}\bigl(\bigl(\delta_U+(\lambda g)\bigr)^*\bigr).$$

Since $(0,-v(P))\in \operatorname{epi}((f+\delta_{\mathcal{A}})^*)$, there is some $\lambda\in C^*$ such that $(0,-v(P))\in \operatorname{epi}(f^*)+\operatorname{epi}((\delta_U+(\bar\lambda g))^*)$. This means that there is some $p\in X^*$ such that $f^*(p)+(\delta_U+(\bar\lambda g))^*(-p)\leqslant -v(P)$, i.e.

$$v(P)\leqslant -f^*(p)-\bigl(\delta_U+(\bar\lambda g)\bigr)^*(-p).$$

Since $-f^*(p)-(\delta_U+(\bar\lambda g))^*(-p)\leqslant -(f+(\delta_U+\bar\lambda g))^*(0)=\inf_{x\in U}[f(x)+(\bar\lambda g)(x)]$, the term in the right-hand side is less than or equal to $v(D)$, which, by weak duality, is less than or equal to $v(P)$. Consequently, the optimal objective value of (D) is attained at $\bar\lambda$ and the necessity is proven. □

Remark 3. One can notice that for some proper convex lower semicontinuous function $f:X\to\overline{\mathbb{R}}$ the concomitant satisfaction of (CC) and $(C(0,\mathcal{A}))$ guarantees the fulfillment of $(C(f,\mathcal{A}))$.

The Farkas–Minkowski property of a system of (infinitely many) convex or linear inequalities has been extensively treated in papers dealing with semi-infinite programming problems, like [9,11,12], and we rediscover it for the set $\mathcal{A}$ as a special case of $(C(0,\mathcal{A}))$.

Remark 4. When T is a possibly infinite index set consider the family of functions $g_t:X\to\overline{\mathbb{R}}$ which are proper, convex and continuous at some point of $\{x\in U:\ g_t(x)\leqslant 0\ \forall t\in T\}$. Take $C=\mathbb{R}^T_+$, denote by $\infty_{\mathbb{R}^T}$ the element attached to $\mathbb{R}^T$ as the greatest with respect to the order induced by the positive cone, and let $(\mathbb{R}^T)^\bullet=\mathbb{R}^T\cup\{\infty_{\mathbb{R}^T}\}$. Consider the function

$$g:X\to\bigl(\mathbb{R}^T\bigr)^\bullet,\quad g(x)=\begin{cases}(g_t(x))_{t\in T}, & \text{if } x\in\bigcap_{t\in T}\operatorname{dom}(g_t),\\ \infty_{\mathbb{R}^T}, & \text{otherwise.}\end{cases}$$

Note that, unlike [11], we do not ask the functions g_t, $t\in T$, to be also lower semicontinuous, which would imply, by Proposition 1.8 in [20], that g is $\mathbb{R}^T_+$-lower semicontinuous. Actually in this setting we note that g need not be even $\mathbb{R}^T_+$-epi-closed. Given these, the condition $(C(0,\mathcal{A}))$ becomes equivalent to saying that $\operatorname{epi}(\sigma_U)+\operatorname{cone}(\bigcup_{t\in T}\operatorname{epi}(g_t^*))$ is closed, which is actually the condition Farkas–Minkowski (FM) in [11]. For each $\lambda\in(\mathbb{R}^T_+)^*$ one has, by Theorem 2.8.7(iii) in [22] and Proposition 2.2 in [4], $\operatorname{epi}((\lambda g)+\delta_U)^*=\operatorname{epi}(\sigma_U)+\sum_{t\in T}\lambda_t\operatorname{epi}(g_t^*)$. Further,

$$\begin{aligned}&\bigcup_{\lambda\in(\mathbb{R}^T_+)^*}\operatorname{epi}\bigl(\bigl(\delta_U+(\lambda g)\bigr)^*\bigr)\\ &\quad=\operatorname{epi}(\sigma_U)+\biggl(\biggl\{\sum_{t\in T'}\lambda_t\operatorname{epi}\bigl(g_t^*\bigr):\ T'\subseteq T,\operatorname{card}(T')<+\infty,\lambda_t>0\ \forall t\in T'\biggr\}\cup\{0\}\times\mathbb{R}_+\biggr)\\ &\quad=\operatorname{cone}\biggl(\biggl(\bigcup_{t\in T}\operatorname{epi}\bigl(g_t^*\bigr)\cup\bigl\{(0,1)\bigr\}\biggr)+\operatorname{epi}(\sigma_U)\biggr)=\operatorname{cone}\biggl(\bigcup_{t\in T}\operatorname{epi}\bigl(g_t^*\bigr)\biggr)+\operatorname{epi}(\sigma_U),\end{aligned}$$

since $\{0\}\times\mathbb{R}_+\subseteq\operatorname{epi}(\sigma_U)$.

Remark 5. Under the hypotheses in Remark 4, from Theorem 2 and Corollary 1 we obtain as special cases and improve the results in Theorem 4.1 in [11] and Theorems 5 and 7 in [9].

4. Characterizations for total Lagrange duality

In this section we deal with another instance of strong duality for an optimization problem and its Lagrange dual, namely the situation when an optimal solution of the primal problem is assumed to be known. We call this situation *total duality*. For any proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ and the set $\mathcal{A}$ we introduce the following condition at $x \in \mathcal{A} \cap \operatorname{dom}(f)$

$$\partial(f + \delta_{\mathcal{A}})(x) = \bigcup_{\substack{\lambda \in C^*, \\ (\lambda g)(x)=0}} \partial\big(f + \delta_U + (\lambda g)\big)(x). \tag{$GBCQ(f,\mathcal{A})$}$$

We say that f and $\mathcal{A}$ satisfy the condition ($GBCQ(f,\mathcal{A})$) when ($GBCQ(f,\mathcal{A})$) is valid for all $x \in \mathcal{A} \cap \operatorname{dom}(f)$.

With this condition we completely characterize the stable total duality for (P) and its Lagrange dual problem (D).

Theorem 3. *Let the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$. $\mathcal{A}$ and f fulfill the condition ($GBCQ(f,\mathcal{A})$) at $\bar{x} \in \mathcal{A} \cap \operatorname{dom}(f)$ if and only if for each $p \in X^*$ for which the infimum over $\mathcal{A}$ of the function $f + \langle p, \cdot\rangle$ is attained at $\bar{x}$ one has*

$$f(\bar{x}) + \langle p, \bar{x}\rangle = \min_{x \in \mathcal{A}}\big[f(x) + \langle p, x\rangle\big] = \max_{\lambda \in C^*} \inf_{x \in U}\big[f(x) + \langle p, x\rangle + (\lambda g)(x)\big]. \tag{1}$$

Proof. Let $\bar{x} \in \mathcal{A} \cap \operatorname{dom}(f)$. For any $p \in X^*$ denote by (P_p) the problem of minimizing $f + \langle p, \cdot\rangle$ over $\mathcal{A}$. We have that $\bar{x}$ is an optimal solution of (P_p) if and only if $0 \in \partial(f + \langle p, \cdot\rangle + \delta_{\mathcal{A}})(\bar{x})$, which is further equivalent to $-p \in \partial(f + \delta_{\mathcal{A}})(\bar{x})$.

"$\Rightarrow$" Let $p \in X^*$ such that $\bar{x}$ solves (P_p). Thus $-p \in \partial(f + \delta_{\mathcal{A}})(\bar{x})$. Because the condition ($GBCQ(f,\mathcal{A})$) is satisfied at $\bar{x}$, there is some $\bar{\lambda} \in C^*$ such that $(\bar{\lambda} g)(\bar{x}) = 0$ and $-p \in \partial(f + \delta_U + (\bar{\lambda} g))(\bar{x})$. The latter means $0 \in \partial(f + \langle p, \cdot\rangle + \delta_U + (\bar{\lambda} g))(\bar{x})$, which leads to

$$f(\bar{x}) + \langle p, \bar{x}\rangle = f(\bar{x}) + \langle p, \bar{x}\rangle + (\bar{\lambda} g)(\bar{x}) = \inf_{x \in U}\big[f(x) + (\bar{\lambda} g)(x) + \langle p, x\rangle\big].$$

Because the inequality

$$\inf_{x \in \mathcal{A}}\big[f(x) + \langle p, x\rangle\big] \geqslant \sup_{\lambda \in C^*} \inf_{x \in U}\big[f(x) + \langle p, x\rangle + (\lambda g)(x)\big]$$

is always fulfilled, we get (1).

"$\Leftarrow$" Let $p \in X^*$ such that $p \in \bigcup_{\substack{\lambda \in C^*, \\ (\lambda g)(\bar{x})=0}} \partial(f + \delta_U + (\lambda g))(\bar{x})$. This means that there is a $\bar{\lambda} \in C^*$ such that $(\bar{\lambda} g)(\bar{x}) = 0$ fulfilling $p \in \partial(f + \delta_U + (\bar{\lambda} g))(\bar{x})$. The latter means actually $0 \in \partial(f - \langle p, \cdot\rangle + \delta_U + (\bar{\lambda} g))(\bar{x})$, i.e. $f(x) - \langle p, x\rangle + \delta_U(x) + (\bar{\lambda} g)(x) \geqslant f(\bar{x}) - \langle p, \bar{x}\rangle + \delta_U(\bar{x}) + (\bar{\lambda} g)(\bar{x})$ $\forall x \in X$. Remember that $\delta_U(\bar{x}) = (\bar{\lambda} g)(\bar{x}) = 0$. As $\delta_{\mathcal{A}}(x) \geqslant \delta_U(x) + (\bar{\lambda} g)(x)$ $\forall x \in X$, we get $f(x) - \langle p, x\rangle + \delta_{\mathcal{A}}(x) \geqslant f(\bar{x}) - \langle p, \bar{x}\rangle + \delta_{\mathcal{A}}(\bar{x})$ $\forall x \in X$. This means actually $p \in \partial(f + \delta_{\mathcal{A}})(\bar{x})$. Thus the inclusion "$\supseteq$" in the expression of ($GBCQ(f,\mathcal{A})$) at $\bar{x}$ is valid.

Take now $p \in \partial(f + \delta_{\mathcal{A}})(\bar{x})$. By the considerations from the beginning of the proof this means that $\bar{x}$ is an optimal solution to (P_{-p}). By (1) there is some $\bar{\lambda} \in C^*$ such that $f(\bar{x}) - \langle p, \bar{x}\rangle = \inf_{x \in U}[f(x) - \langle p, x\rangle + (\bar{\lambda} g)(x)]$. As the infimum in the right-hand side is less than or equal to $f(\bar{x}) - \langle p, \bar{x}\rangle + (\bar{\lambda} g)(\bar{x})$, we get that $(\bar{\lambda} g)(\bar{x}) \geqslant 0$. Because $\bar{x} \in \mathcal{A}$ and $\bar{\lambda} \in C^*$ we have $(\bar{\lambda} g)(\bar{x}) \leqslant 0$, thus $(\bar{\lambda} g)(\bar{x}) = 0$. We have

$$f(\bar{x}) - \langle p, \bar{x}\rangle + (\bar{\lambda} g)(\bar{x}) = \inf_{x \in U}\big[f(x) + (\bar{\lambda} g)(x) - \langle p, x\rangle\big],$$

which leads to $0 \in \partial(f + (\bar{\lambda} g) + \delta_U - \langle p, \cdot\rangle)(\bar{x})$, i.e. $p \in \partial(f + \delta_U + (\bar{\lambda} g))(\bar{x})$. This yields $p \in \bigcup_{\substack{\lambda \in C^*, \\ (\lambda g)(\bar{x})=0}} \partial(f + \delta_U + (\lambda g))(\bar{x})$, i.e. the inclusion "$\subseteq$" in the expression of ($GBCQ(f,\mathcal{A})$) is fulfilled at $\bar{x}$, too. Therefore ($GBCQ(f,\mathcal{A})$) holds at $\bar{x}$. $\square$

The following statement follows naturally.

Theorem 4. *Let the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$. $\mathcal{A}$ and f fulfill the condition $(GBCQ(f, \mathcal{A}))$ if and only if for each $p \in X^*$ for which the infimum over $\mathcal{A}$ of the function $f + \langle p, \cdot \rangle$ is attained one has*

$$\min_{x \in \mathcal{A}} \big[f(x) + \langle p, x \rangle \big] = \max_{\lambda \in C^*} \inf_{x \in U} \big[f(x) + \langle p, x \rangle + (\lambda g)(x) \big].$$

When $f(x) = 0$ $\forall x \in X$, $(GBCQ(f, \mathcal{A}))$ turns into a condition which generalizes the classical basic constraint qualification at $x \in \mathcal{A}$

$$\bigcup_{\substack{\lambda \in C^*, \\ (\lambda g)(x) = 0}} \partial \big(\delta_U + (\lambda g) \big)(x) = \partial \delta_{\mathcal{A}}(x). \tag{$GBCQ(0, \mathcal{A})$}$$

If the set $\mathcal{A}$ satisfies the condition $(GBCQ(0, \mathcal{A}))$ for all $x \in \mathcal{A}$ we say that it fulfills the condition $(GBCQ(0, \mathcal{A}))$.

A direct consequence of Theorem 3 is the next result, where the condition $(GBCQ(0, \mathcal{A}))$ at some $\bar{x} \in \mathcal{A}$ completely characterizes the total Lagrange duality for optimization problems consisting in minimizing linear functionals that attain their minimum over $\mathcal{A}$ at $\bar{x}$.

Corollary 2. *$\mathcal{A}$ fulfills the condition $(GBCQ(0, \mathcal{A}))$ at $\bar{x} \in \mathcal{A}$ if and only if for each $p \in X^*$ such that $\langle p, \cdot \rangle$ attains its minimum over $\mathcal{A}$ at $\bar{x}$ one has*

$$\langle p, \bar{x} \rangle = \min_{x \in \mathcal{A}} \langle p, x \rangle = \max_{\lambda \in C^*} \inf_{x \in U} \big[\langle p, x \rangle + (\lambda g)(x) \big].$$

The next theorem completely characterizes via $(GBCQ(0, \mathcal{A}))$ at some $\bar{x} \in \mathcal{A}$ the strong duality for convex optimization problems consisting in minimizing over the set $\mathcal{A}$ of proper convex lower semicontinuous functions $f : X \to \overline{\mathbb{R}}$ which attain their minima over $\mathcal{A}$ at $\bar{x}$ and fulfill the following condition (see [4])

$$f^* \,\square\, \delta^*_{\mathcal{A}} \quad \text{is a lower semicontinuous function and it is exact at } 0, \tag{FRC}$$

and their Lagrange dual problems.

Remark 6. The condition (FRC) is weaker than (CC) and in [4] there is an example that shows that it is possible to have the first of them fulfilled and the second violated. Consequently, if one removes the assumption of lower semicontinuity from f and takes it continuous at some point of $\mathcal{A}$, then condition (FRC) is automatically satisfied.

Theorem 5. *$\mathcal{A}$ fulfills the condition $(GBCQ(0, \mathcal{A}))$ at $\bar{x} \in \mathcal{A}$ if and only if for each proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ that fulfills $\mathcal{A} \cap \operatorname{dom}(f) \neq \emptyset$ and attains its minimum over $\mathcal{A}$ at $\bar{x}$ and satisfies (FRC) one has*

$$f(\bar{x}) = \inf_{x \in \mathcal{A}} f(x) = \max_{\lambda \in C^*} \inf_{x \in U} \big[f(x) + (\lambda g)(x) \big].$$

Proof. As the sufficiency follows obviously from the preceding theorem by taking f linear, we prove here only the necessity. Take some f as requested in the hypothesis. We have

$$f(\bar{x}) = \inf_{x \in \mathcal{A}} f(x) = -(f + \delta_{\mathcal{A}})^*(0)$$

and (FRC) guarantees (cf. [4]) that there is some $p \in X^*$ such that $(f + \delta_{\mathcal{A}})^*(0) = f^*(p) + \sigma_{\mathcal{A}}(-p)$. Further we get

$$0 = f(\bar{x}) + f^*(p) + \sigma_{\mathcal{A}}(-p) + \delta_{\mathcal{A}}(\bar{x}) \geqslant \langle p, \bar{x} \rangle + \langle -p, \bar{x} \rangle = 0,$$

therefore there are equalities in Young–Fenchel's inequality for both pairs f and f^*, and $\delta_{\mathcal{A}}$ and $\sigma_{\mathcal{A}}$, respectively, i.e. $p \in \partial f(\bar{x})$ and $-p \in \partial \delta_{\mathcal{A}}(\bar{x})$. By $(GBCQ(0, \mathcal{A}))$ at $\bar{x}$ there is a $\bar{\lambda} \in C^*$ such that $(\bar{\lambda} g)(\bar{x}) = 0$ and $-p \in \partial(\delta_U + (\bar{\lambda} g))(\bar{x})$. Consequently, $(\delta_U + (\bar{\lambda} g))(\bar{x}) + (\delta_U + (\bar{\lambda} g))^*(-p) = \langle -p, \bar{x} \rangle$ and this yields $f(\bar{x}) + f^*(p) + (\delta_U + (\bar{\lambda} g))^*(-p) = 0$. Further,

$$f(\bar{x}) = -f^*(p) - \big(\delta_U + (\bar{\lambda} g) \big)^*(-p) \leqslant -\big(f + \delta_U + (\bar{\lambda} g) \big)^*(0) = \inf_{x \in U} \big[f(x) + (\bar{\lambda} g)(x) \big] \leqslant \inf_{x \in \mathcal{A}} f(x) = f(\bar{x}),$$

and the proof is completed. □

Remark 7. Using Remark 4.2 in [5], one can prove that Theorem 5 can be proven in a more general context, namely by considering that the functions f satisfy instead of (*FRC*) the condition $f^* \,\square\, \delta^*_{\mathcal{A}}$ is lower semicontinuous at 0 and it is exact at 0.

Such statements are valid also for the condition ($GBCQ(0, \mathcal{A})$) as follows.

Theorem 6. *The following statements are equivalent*:

(i) *$\mathcal{A}$ fulfills the condition* ($GBCQ(0, \mathcal{A})$),

(ii) *for each $p \in X^*$ that attains its minimum over $\mathcal{A}$ one has*

$$\min_{x\in\mathcal{A}} \langle p, x\rangle = \max_{\lambda\in C^*} \inf_{x\in U} \big[\langle p, x\rangle + (\lambda g)(x)\big],$$

(iii) *for each proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ that fulfills $\mathcal{A} \cap \operatorname{dom}(f) \neq \emptyset$ and attains its minimum over $\mathcal{A}$ and satisfies* (*FRC*) *one has*

$$\min_{x\in\mathcal{A}} f(x) = \max_{\lambda\in C^*} \inf_{x\in U} \big[f(x) + (\lambda g)(x)\big].$$

Remark 8. When g is continuous at some point of $\mathcal{A}$, the condition ($GBCQ(0, \mathcal{A})$) turns at each $x \in \mathcal{A}$ into

$$\partial\delta_U(x) + \bigcup_{\substack{\lambda\in C^*,\\ (\lambda g)(x)=0}} \partial(\lambda g)(x) = \partial\delta_{\mathcal{A}}(x).$$

Remark 9. Let T be a possibly infinite index set and let g be as in Remark 4. In this setting the condition ($GBCQ(0, \mathcal{A})$) at x becomes the so-called locally Farkas–Minkowski condition at x (cf. [9,10])

$$\partial\delta_U(x) + \operatorname{cone}\bigg(\bigcup_{t\in T(x)} \partial g_t(x)\bigg) = \partial\delta_{\mathcal{A}}(x), \tag{LFM}$$

where $T(x) = \{t \in T\colon\ g_t(x) = 0\}$, which is known also under the name (*BCQ*) at x (cf. [9]). In this case ($GBCQ(0, \mathcal{A})$) becomes exactly the condition (*LFM*) in [11]. By Theorem 2.8.7(iii) in [22] ($GBCQ(0, \mathcal{A})$) turns into

$$\partial\delta_{\mathcal{A}}(x) = \partial\delta_U(x) + \bigcup_{\substack{\lambda\in(\mathbb{R}^T_+)^*,\\ \lambda=(\lambda_t)_{t\in T},\\ \sum_{t\in T}\lambda_t g_t(x)=0}} \partial\bigg(\sum_{t\in T}\lambda_t g_t\bigg)(x).$$

Further,

$$\begin{aligned}\bigcup_{\substack{\lambda\in(\mathbb{R}^T_+)^*,\\ \lambda=(\lambda_t)_{t\in T},\\ \sum_{t\in T}\lambda_t g_t(x)=0}} \partial\bigg(\sum_{t\in T}\lambda_t g_t\bigg)(x) &= \bigg\{\sum_{t\in T'}\lambda_t\partial g_t(x)\colon\ T' \subseteq T,\ \operatorname{card}(T') < +\infty,\ \lambda_t > 0,\ g_t(x) = 0\ \forall t \in T'\bigg\} \cup \{0\}\\ &= \operatorname{cone}\bigg(\bigcup_{t\in T(x)} \partial g_t(x)\bigg),\end{aligned}$$

and adding the set in the right-hand side to $\partial\delta_U(x)$, what we obtain is actually $\partial\delta_U(x) + \operatorname{cone}(\bigcup_{t\in T(x)} \partial g_t(x))$. If T is a finite index set and $U = X$, ($GBCQ(0, \mathcal{A})$) is actually the condition (*BCQ*) considered in [21]. Moreover, if T contains only one element, i.e. $g : X \to \overline{\mathbb{R}}$, when $C = \mathbb{R}_+$ ($GBCQ(0, \mathcal{A})$) is actually the condition (5) in [21], while when $U = X$ and $x \in \operatorname{bd}(\mathcal{A})$, ($GBCQ(0, \mathcal{A})$) at x becomes the condition (*BCQ*) at x in [14]. Considering $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, the convex functions $c_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \dots, r$, and $\mathcal{A} = \{x \in \mathbb{R}^n\colon\ Ax = b, c_j(x) \leqslant 0,\ j = 1, \dots, r\}$, ($GBCQ(0, \mathcal{A})$) becomes exactly the condition (*BCQ*) in its original formulation due to Hiriart-Urruty and Lemaréchal [13]. For comparisons between other constraint qualifications and different particular instances of (*BCQ*) we refer to [14,17,18,21].

Remark 10. When T is a possibly infinite index set and $g=(g_t)_{t\in T}$ such that each g_t, $t\in T$, is continuous at some point of $\mathcal{A}$ and $C=(\mathbb{R}_+^T)^*$, Theorem 6 yields, via Remark 9, a result similar to Theorem 5.1 in [11], improving it because the functions g_t, $t\in T$, are no more required to be lower semicontinuous as there and also in the sense that (ii) in the mentioned statement can be generalized by taking f not continuous at some point of $\mathcal{A}\cap \operatorname{dom}(f)$ like in the original paper, but only fulfilling the condition (*FRC*) or the weaker condition mentioned in Remark 7. Moreover, if T contains only one element, and when $C=\mathbb{R}_+$, Theorem 6 generalizes Proposition 2.5 in [21].

Remark 11. By Theorems 1 and 3 one can easily notice that $(C(f,\mathcal{A}))$ implies $(GBCQ(f,\mathcal{A}))$, so we also have that $(C(0,\mathcal{A}))$ guarantees the fulfillment of $(GBCQ(0,\mathcal{A}))$. This generalizes Corollary 2 in [9]. See Example 4.1 in [11] for a situation when $(GBCQ(0,\mathcal{A}))$ is valid, while $(C(0,\mathcal{A}))$ fails.

Remark 12. As one could notice in the proofs of Theorem 2 and Theorem 5, their hypotheses (and also the ones of Theorem 6) ensure also that

$$\inf_{x\in U}\big[f(x)+(\lambda g)(x)\big]=\max_{\beta\in X^*}\big\{-f^*(\beta)-(\lambda g)_U^*(-\beta)\big\},$$

thus the optimal value of the Lagrange dual problem (D) is equal in each case to the optimal value of the Fenchel–Lagrange dual to (P) (cf. [3,5])

$$\sup_{\substack{\lambda\in C^*,\\ \beta\in X^*}}\big\{-f^*(\beta)-(\lambda g)_U^*(-\beta)\big\}. \tag{$\overline{D}$}$$

In [3] we completely characterized via a regularity condition the stable strong duality for the problems (P) and $(\overline{D})$.

We conclude this section by giving optimality conditions for the problem (P).

Theorem 7. *If $\mathcal{A}$ fulfills the condition $(C(0,\mathcal{A}))$ and $f:X\to\overline{\mathbb{R}}$ is a proper convex lower semicontinuous function which satisfies (CC), $\bar{x}\in\mathcal{A}\cap\operatorname{dom}(f)$ is an optimal solution to (P) if and only if there is some $\bar{\lambda}\in C^*$ such that $(\bar{\lambda}g)(\bar{x})=0$ and $0\in\partial f(\bar{x})+\partial(\delta_U+(\bar{\lambda}g))(\bar{x})$.*

Proof. Since $(C(0,\mathcal{A}))$ holds, by Remark 11 one has

$$\partial\delta_{\mathcal{A}}(x)=\bigcup_{\substack{\lambda\in C^*,\\ (\lambda g)(x)=0}}\partial\big(\delta_U+(\lambda g)\big)(x)\quad \forall x\in\mathcal{A}.$$

We know that $\bar{x}\in\mathcal{A}\cap\operatorname{dom}(f)$ is an optimal solution to (P) if and only if $0\in\partial(f+\delta_{\mathcal{A}})(\bar{x})$. Because of (*CC*), by Theorem 3.2 in [4] this is further equivalent to $0\in\partial f(\bar{x})+\partial\delta_{\mathcal{A}}(\bar{x})$, i.e.

$$0\in\partial f(\bar{x})+\bigcup_{\substack{\lambda\in C^*,\\ (\lambda g)(\bar{x})=0}}\partial\big(\delta_U+(\lambda g)\big)(\bar{x}),$$

thus the equivalence in the conclusion follows. □

Remark 13. The theorem remains valid if we weaken the hypotheses by taking $\mathcal{A}$ to fulfill only $(GBCQ(0,\mathcal{A}))$, not $(C(0,\mathcal{A}))$.

Remark 14. Note that when $(C(0,\mathcal{A}))$ holds, (*CC*) is equivalent to saying that $\operatorname{epi}(f^*)+\bigcup_{\lambda\in C^*}\operatorname{epi}((\lambda g)+\delta_U)^*$ is closed. For the special case when g is continuous, by Theorem 7 one obtains the results in Theorem 4.2 in [6] and Theorem 5.5 in [8]. If moreover f is continuous, by Theorem 7 we obtain Corollary 3.2 in [15].

5. Conclusions

We completely characterize the *strong* and *stable strong* Lagrange duality for a convex optimization problem through equivalent conditions. Then we introduce necessary and sufficient conditions which characterize the strong and stable strong Lagrange duality for the case when a solution of the primal problem is assumed to exist, situations called by us *total*, respectively *stable total Lagrange duality*. The conditions we use extend the so-called Farkas–Minkowski and locally Farkas–Minkowski conditions given so far for convex optimization problems having infinitely many convex inequalities as constraints. Different results in the literature are also rediscovered as special cases and some of them are improved in their original context.

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